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# REGULAR, PARTIALLY INVARIANT SOLUTIONS OF RANK 1 AND DEFECT 1 OF EQUATIONS OF PLANE MOTION OF A VISCOUS HEAT-CONDUCTING GAS

**V. V. Bublik**

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*A system of the Navier–Stokes equations of two-dimensional motion of a viscous heat-conducting perfect gas with a polytropic equation of state is considered. Regular, partially invariant solutions of rank 1 and defect 1 are studied. A sufficient condition of their reducibility to invariant solutions of rank 1 is proved. All solutions of this class with a linear dependence of the velocity-vector components on spatial coordinates are examined. New examples of solutions that are not reducible to invariant solutions are obtained.*

**Key words:** *dynamics of a viscous heat-conducting gas, partially invariant solutions.*

**1. Descriptions of the Model.** We consider a system of equations that describe plane motion of a viscous heat-conducting perfect gas with a polytropic equation of state:

$$\rho(u_t + uu_x + vu_y) = -p_x + (\lambda(u_x + v_y))_x + (2\mu u_x)_x + (\mu(u_y + v_x))_y; \tag{1.1}$$

$$\rho(v_t + uv_x + vv_y) = -p_y + (\lambda(u_x + v_y))_y + (\mu(u_y + v_x))_x + (2\mu v_y)_y; \tag{1.2}$$

$$\rho_t + (u\rho)_x + (v\rho)_y = 0; \tag{1.3}$$

$$c_V\rho(T_t + uT_x + vT_y) + p(u_x + v_y) = (\varkappa T_x)_x + (\varkappa T_y)_y + \lambda(u_x + v_y)^2 + \mu(2u_x^2 + 2v_y^2 + (u_y + v_x)^2). \tag{1.4}$$

Here $u$ and $v$ are the components of the velocity vector, $\rho$ is the density, $T$ is the temperature, $p = R\rho T$ is the pressure, $R$ is the gas constant, $c_V$ is the specific heat at constant volume, $\mu = m_0T^\omega$ and $\lambda = l_0T^\omega$ are the first and second viscosities, and $\varkappa = k_0T^\omega$ is the thermal conductivity. In our study, we take into account the following physically meaningful conditions:

$$\rho > 0, \quad T > 0, \quad 3\lambda + 2\mu \geq 0, \quad \mu > 0, \quad \varkappa \geq 0, \quad R > 0, \quad c_V > 0. \tag{1.5}$$

A group classification of system (1.1)–(1.4) given in [1] for the case $\lambda = -(2/3)\mu$ is also valid if there is no such a dependence between the first and second viscosities. The group admitted by system (1.1)–(1.4) is an eight-parameter group. This group corresponds to the Lie algebra $L_8$ with the basis

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = t\partial_x + \partial_u, \quad X_4 = t\partial_y + \partial_v,$$

$$X_5 = y\partial_x - x\partial_y + v\partial_u - u\partial_v, \quad X_6 = \partial_t, \quad X_7 = t\partial_t + x\partial_x + y\partial_y - \rho\partial_\rho,$$

$$X_8 = x\partial_x + y\partial_y + u\partial_u + v\partial_v + 2(\omega - 1)\rho\partial_\rho + 2T\partial_T.$$

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Khristianovich Institute of Theoretical and Applied Mechanics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090; bublik@itam.nsc.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 47, No. 6, pp. 23–33, November–December, 2006. Original article submitted June 15, 2005; revision submitted December 12, 2005.

 

TABLE 1

| Subgroup number | $H$ | $J$ |
|---|---|---|
| 1 | 1, 3, $7+\alpha 8$ | $yt^{-\alpha-1}$, $vt^{-\alpha}$, $\rho t^{1-2\alpha(\omega-1)}$, $Tt^{-2\alpha}$ |
| 2 | 1, 3, $2+7-8$ | $y-\ln t$, $vt$, $\rho t^{2\omega-1}$, $Tt^2$ |
| 3 | 1, 3, $4+7$ | $y/t-\ln t$, $v-\ln t$, $\rho t$, $T$ |
| 4 | 1, 3, $4+6$ | $t^2-2y$, $v-t$, $\rho$, $T$ |
| 5 | 1, 3, $6+8$ | $y\,\mathrm{e}^{-t}$, $v\,\mathrm{e}^{-t}$, $\rho\mathrm{e}^{2(1-\omega)t}$, $T\mathrm{e}^{-2t}$ |
| 6 | 1, 3, $6-8$ | $y\mathrm{e}^{t}$, $v\mathrm{e}^{t}$, $\rho\mathrm{e}^{2(\omega-1)t}$, $T\mathrm{e}^{2t}$ |
| 7 | 1, 3, 6 | $y$, $v$, $\rho$, $T$ |
| 8 | 1, 3, 8 | $t$, $v/y$, $\rho y^{2(1-\omega)}$, $Ty^{-2}$ |
| 9 | 1, 3, 4 | $t$, $tv-y$, $\rho$, $T$ |
| 10 | 1, 2, 3 | $t$, $v$, $\rho$, $T$ |

The optimal system of subalgebras of the Lie algebra $L_8$ was constructed in [2]. We consider solutions that can be constructed on the basis of three-dimensional subalgebras. In this case, the simplest class of solutions consists of invariant solutions of rank 0. All solutions of this kind for system (1.1)–(1.4) are described in [3]. A class of solutions whose description involves more difficulties is the class of partially invariant solutions. The notion of a regular, partially invariant solution was introduced in [4]. The goal of the present work is to study regular, partially invariant solutions of rank 1 and defect 1 of system (1.1)–(1.4), which can be constructed on the basis of three-dimensional subalgebras. In particular, a sufficient condition of their reducibility to invariant solutions of rank 1 described in [2] is proved. All solutions of this class with a linear dependence of the velocity-vector components on spatial coordinates are also considered.

**2. Partially Invariant Solutions Reducible to Invariant Solutions.** Invariant solutions are much more readily constructed than partially invariant solutions. Using criteria that allow one to eliminate invariant solutions reducible to partially invariant solutions beforehand, one can construct nonreducible solutions directly. A criterion of reduction of an arbitrary partially invariant manifold to an invariant manifold was proved in [5]. It is not always convenient, however, to apply this criterion in practice. Therefore, it is important to prove particular theorems formulating sufficient conditions. The known Ovsyannikov's theorem [6, Theorem 22.7] is inapplicable in most cases to systems of differential equations of the second order. In the present work, we consider the issue of reducibility of regular, partially invariant solutions of rank 1 and defect 1 of equations of plane motion of a viscous heat-conducting perfect gas. A similar problem for axisymmetric motion of the gas was considered in [7].

As there is a correspondence between the Lie algebras and the Lie groups of transformations, we further use both notions without specific comments on the area of applicability of each notion. The group of transformations corresponding to the Lie algebra $L_8$ is denoted by $G_8$.

**Theorem 1.** *If the universal invariant of the subgroup $H \subset G_8$ can be chosen in the form*

$$J = (\xi(t,x,y), A(t,x,y)u + B(t,x,y)v + C(t,x,y), D(t,x,y)\rho, E(t,x,y)T), \tag{2.1}$$

*where $\xi$, $A$, $B$, $C$, $D$, and $E$ are some functions, the corresponding regular, partially invariant $H$-solution of rank 1 and defect 1 of system* (1.1)–(1.4) *is reducible to an invariant solution.*

**Proof.** It is known that the rank and defect of a partially invariant solution are invariant with respect to a similarity transformation of subgroups. An analysis of invariants of all subgroups shows that satisfaction of condition (2.1) for each subgroup is also invariant with respect to a similarity transformation of subgroups. Therefore, it is sufficient to prove the theorem only for subgroups from the optimal system of subgroups. Table 1 contains all subgroups that satisfy condition (2.1). For convenience, subgroup $H$ is identified by the basis of the corresponding Lie algebra (only the operator number is indicated instead of the operator itself; for instance, $7+\alpha 8$ indicates the operator $X_7+\alpha X_8$). Subgroup invariants are listed in column $J$.

TABLE 2

| Subgroup number | $H$ | $j_u$ |
|---|---|---|
| 1 | $1,\ -\alpha\beta 3+7+\alpha 8\ (\alpha\neq 0)$ | $(u-\beta)t^{-\alpha}$ |
| | $1,\ \alpha 3+7$ | $u-\alpha\ln t$ |
| | $3,\ (\alpha+1)\beta 1+7+\alpha 8\ (\alpha\neq -1)$ | $(ut-x-\beta)t^{-\alpha-1}$ |
| | $3,\ -\alpha 1+7-8$ | $ut-x-\alpha\ln t$ |
| 2 | $1,\ \alpha 3+2+7-8$ | $(u-\alpha)t$ |
| | $3,\ -\alpha 1+2+7-8$ | $ut-x-\alpha\ln t$ |
| 3 | $1,\ \alpha 3+4+7$ | $u-\alpha\ln t$ |
| | $3,\ \alpha 1+4+7$ | $(tu-x-\alpha)/t$ |
| 4 | $1,\ \alpha 3+4+6$ | $u-\alpha t$ |
| 5 | $1,\ -\alpha 3+6+8$ | $(u-\alpha)\,\mathrm{e}^{-t}$ |
| 6 | $1,\ -\alpha 3+6-8$ | $(u-\alpha)\,\mathrm{e}^{t}$ |
| 7 | $1,\ \alpha 3+6$ | $u-\alpha t$ |
| 8 | $\alpha 1+3,\ \beta 3+8$ | $((t+\alpha)u-x-\beta)/y$ |
| 9 | $\alpha 1+3,\ -\beta 1+4$ | $t(t+\alpha)u-tx-\beta y$ |
| 10 | $\beta 1-2,\ \alpha 1+3$ | $(t+\alpha)u-x-\beta y$ |

We present the general scheme of the theorem proof. For each subgroup in Table 1, the invariants define the form of solution presentation. For $\partial\xi/\partial y\neq 0$, Eq. (1.3) yields the expression

$$u(t,x,y)=\varphi(\xi)x/t+w(t,\xi) \tag{2.2}$$

or

$$u(t,x,y)=\varphi(\xi)x+w(t,\xi). \tag{2.3}$$

Then, Eq. (1.4) takes the form

$$F(\xi)+\Big(f_1(t)\varphi' x+f_2(t)\,\frac{\partial w}{\partial\xi}\Big)^2=0, \tag{2.4}$$

where $F$, $f_1$, and $f_2$ are known (specific for each subgroup) functions of the indicated arguments. It follows from Eq. (2.4) that

$$\varphi'=0,\qquad 2f_2\,\frac{\partial w}{\partial\xi}\Big(f_2'\,\frac{\partial w}{\partial\xi}+f_2\,\frac{\partial^2 w}{\partial t\,\partial\xi}\Big)=0. \tag{2.5}$$

For $\xi=t$, we obtain $u(t,x,y)=\varphi(t)x+w(t,y)$ instead of (2.2) or (2.3) and $F(t)+(\partial w/\partial y)^2=0$ instead of (2.4), which yields

$$2\,\frac{\partial w}{\partial y}\,\frac{\partial^2 w}{\partial y^2}=0. \tag{2.6}$$

The integrals of Eqs. (2.5) or (2.6) are used to analyze Eq. (1.1). As a result, we obtain a presentation for $u$, which allows us to find a two-dimensional subgroup with respect to which the solution constructed is invariant. All such subgroups are listed in Table 2. The first column indicates the subgroup number from Table 1, and the second column contains the bases of the corresponding two-dimensional subgroups. The constants can take all real values, except for specially indicated cases. The third column shows an additional invariant of subgroup $H$, which is absent for the corresponding three-dimensional subgroup (all other invariants coincide). The proof for each subgroup is not given here.

**3. Solutions with a Linear Velocity Field.** There are eight series of three-dimensional subgroups that do not satisfy the necessary condition of existence of the invariant solution or the condition of the above-described theorem. All of them are listed in Table 3. If regular, partially invariant solutions of rank 1 and defect 1

TABLE 3

| Subgroup number | $H$ | $J$ |
|---|---|---|
| 1 | 1, 2, 8 | $t,\ v/u,\ \rho u^{2(1-\omega)},\ Tu^{-2}$ |
| 2 | $\alpha 1+2$, 3, 8 | $t,\ (tu-x+\alpha y)/v,\ \rho v^{2(1-\omega)},\ Tv^{-2}$ |
| 3 | 1, 2, 5 | $t,\ u^2+v^2,\ \rho,\ T$ |
| 4 | 3, 4, 5 | $t,\ (ut-x)^2+(vt-y)^2,\ \rho,\ T$ |
| 5 | 1, 2, $5+\alpha 8$ ($\alpha\neq 0$) | $\omega\neq 1$: $t,\ \left(u\sin\frac{\ln\rho}{2\alpha(\omega-1)}+v\cos\frac{\ln\rho}{2\alpha(\omega-1)}\right)\rho^{1/(2-2\omega)},$ $\left(u\cos\frac{\ln\rho}{2\alpha(\omega-1)}-v\sin\frac{\ln\rho}{2\alpha(\omega-1)}\right)\rho^{1/(2-2\omega)},\ T\rho^{1/(1-\omega)}$ |
| | | $\omega=1$: $t,\ \left(u\sin\frac{\ln T}{2\alpha}+v\cos\frac{\ln T}{2\alpha}\right)T^{-1/2},$ $\left(u\cos\frac{\ln T}{2\alpha}-v\sin\frac{\ln T}{2\alpha}\right)T^{-1/2},\ \rho$ |
| 6 | 3, 4, $5+\alpha 8$ ($\alpha\neq 0$) | $\omega\neq 1$: $\left((ut-x)\sin\frac{\ln\rho}{2\alpha(\omega-1)}+(vt-y)\cos\frac{\ln\rho}{2\alpha(\omega-1)}\right)\rho^{1/(2-2\omega)},$ $\left((ut-x)\cos\frac{\ln\rho}{2\alpha(\omega-1)}-(vt-y)\sin\frac{\ln\rho}{2\alpha(\omega-1)}\right)\rho^{1/(2-2\omega)},$ $t,\ T\rho^{1/(1-\omega)}$ |
| | | $\omega=1$: $t,\ \left((ut-x)\sin\frac{\ln T}{2\alpha}+(vt-y)\cos\frac{\ln T}{2\alpha}\right)T^{-1/2},$ $\left((ut-x)\cos\frac{\ln T}{2\alpha}-(vt-y)\sin\frac{\ln T}{2\alpha}\right)T^{-1/2},\ \rho$ |
| 7 | 1, 6, $7-8$ | $y,\ v/u,\ \rho u^{1-2\omega},\ Tu^{-2}$ |
| 8 | $5+\alpha 8$, 6, $7-8$ | $\sqrt{x^2+y^2}\exp(\alpha\arctan(y/x)),\ (ux+vy)/(vx-uy),$ $\sqrt{x^2+y^2}\,(u^2+v^2)^{1/2-\omega}\rho,\ T(x^2+y^2)/(ux+vy)^2$ |

reducible to invariant solutions do exist, they can be constructed only on the basis of subgroups listed in Table 3. A comprehensive study of such solutions requires the use of the theories of compatibility analysis of systems of partial derivatives [8, 9]. Such an analysis has not been performed for the above-mentioned eight subgroups, and we confine ourselves to a particular case, namely, to solutions with a linear dependence of the velocity field on spatial coordinates (we will use the term "linear" for such a velocity field for brevity). This kind of motion of continuous media has been considered in numerous papers (see, e.g., a brief review in [10]). Moreover, a more general class of solutions has been studied, namely, solutions with a linear dependence of the velocity field on some of the spatial coordinates (see [11] and the references therein).

*Subgroup* 1. The solution is presented as

$$u=u(t,x,y),\qquad v=v_1(t)u,\qquad \rho=\rho_1(t)u^{2(\omega-1)},\qquad T=T_1(t)u^2. \tag{3.1}$$

The condition of velocity-field linearity is

$$u=u_1(t)x+u_2(t)y+u_3(t). \tag{3.2}$$

Substituting Eqs. (3.1) and (3.2) into system (1.1)–(1.4) and splitting the latter with respect to the variables $x$ and $y$, we obtain a system of the form

$$\frac{du_1}{u_1} = \frac{du_2}{u_2} = \frac{du_3}{u_3} = U\,dt, \qquad \frac{dv_1}{dt} = V, \qquad \frac{d\rho_1}{dt} = \varrho, \qquad \frac{dT_1}{dt} = \Theta, \tag{3.3}$$

where $U$, $V$, $\varrho$, and $\Theta$ are known functions of $u_1$, $u_2$, $u_3$, $v_1$, $\rho_1$, and $T_1$. Ovsyannikov's theorem predicts that the solution of system (3.3) is reducible to an invariant solution. A particular form of the subgroup with respect to which the solution is invariant is determined after the constants of integration of system (3.3) are prescribed.

*Subgroup* 2. The solution is presented as

$$u = (vu_1(t) + x - \alpha y)/t, \qquad v = v(t, x, y), \qquad \rho = \rho_1(t) v^{2(\omega-1)}, \qquad T = T_1(t) v^2. \tag{3.4}$$

The condition of velocity-field linearity is

$$v = v_1(t)x + v_2(t)y + v_3(t). \tag{3.5}$$

Substituting Eqs. (3.4) and (3.5) into system (1.1)–(1.4) and splitting the latter with respect to the variables $x$ and $y$, we obtain a system of the form

$$\frac{du_1}{dt} = U, \qquad \frac{dv_1}{dt} + \frac{u_1 v_1 + 1}{t}\, v_1 = v_1 V, \qquad \frac{dv_2}{dt} + \frac{u_1 v_2 - \alpha}{t}\, v_1 = v_2 V,$$

$$\frac{dv_3}{dt} + \frac{u_1 v_3}{t}\, v_1 = v_3 V, \qquad \frac{d\rho_1}{dt} = \varrho, \qquad \frac{dT_1}{dt} = \Theta,$$

where $U$, $V$, $\varrho$, and $\Theta$ are known functions of $u_1$, $v_1$, $v_2$, $v_3$, $\rho_1$, and $T_1$. Ovsyannikov's theorem predicts that the solution is reducible to an invariant solution.

*Subgroup* 3. The solution is presented as

$$u^2 + v^2 = q^2(t), \qquad \rho = \rho(t), \qquad T = T(t).$$

The linear velocity field is described as

$$u = u_1(t)x + u_2(t)y + u_3(t), \qquad v = v_1(t)x + v_2(t)y + v_3(t).$$

We have

$$(u_1 x + u_2 y + u_3)^2 + (v_1 x + v_2 y + v_3)^2 = q^2,$$

which yields, after splitting with respect to $x$ and $y$,

$$u_1 \equiv 0, \qquad v_1 \equiv 0, \qquad u_2 \equiv 0, \qquad v_2 \equiv 0, \qquad u_3^2 + v_3^2 = q^2.$$

The solution obtained is invariant with respect to the subgroup $\{X_1, X_2\}$. Reducibility is proved.

*Subgroup* 4. The solution is presented as

$$(ut - x)^2 + (vt - y)^2 = q^2(t), \qquad \rho = \rho(t), \qquad T = T(t). \tag{3.6}$$

The linear velocity field is described as

$$u = u_1(t)x + u_2(t)y + u_3(t), \qquad v = v_1(t)x + v_2(t)y + v_3(t). \tag{3.7}$$

Substituting Eq. (3.7) into the first equation in (3.6) and splitting it with respect to $x$ and $y$, we obtain

$$u_1 t - 1 = 0, \qquad v_2 t - 1 = 0, \qquad u_2 \equiv 0, \qquad v_1 \equiv 0, \qquad (u_3^2 + v_3^2)t^2 = q^2.$$

The solution obtained is invariant with respect to the subgroup $\{X_3, X_4\}$. Reducibility is proved.

*Subgroup* 5 ($\omega \neq 1$). The solution is presented as

$$u = \rho^{1/(2\omega-2)}\Big(\varphi(t)\sin\frac{\ln\rho}{2\alpha(\omega-1)} + \psi(t)\cos\frac{\ln\rho}{2\alpha(\omega-1)}\Big) \equiv U(t,\rho),$$

$$v = \rho^{1/(2\omega-2)}\Big(\varphi(t)\cos\frac{\ln\rho}{2\alpha(\omega-1)} - \psi(t)\sin\frac{\ln\rho}{2\alpha(\omega-1)}\Big) \equiv V(t,\rho),$$

$$T = T_1(t)\rho^{1/(\omega-1)}, \qquad \rho = \rho(t, x, y).$$

The linear velocity field is described as

$$u_{xx} = u_{xy} = u_{yy} = v_{xx} = v_{xy} = v_{yy} = 0. \tag{3.8}$$

As we have

$$\frac{\partial U}{\partial \rho} = \frac{\alpha U + V}{2\alpha(\omega - 1)\rho}, \qquad \frac{\partial V}{\partial \rho} = \frac{\alpha V - U}{2\alpha(\omega - 1)\rho},$$

system (3.8) acquires the form

$$\begin{gathered} A\rho\rho_{xx} + B\rho_x^2 = 0, \qquad A\rho\rho_{xy} + B\rho_x\rho_y = 0, \qquad A\rho\rho_{yy} + B\rho_y^2 = 0, \\ C\rho\rho_{xx} + D\rho_x^2 = 0, \qquad C\rho\rho_{xy} + D\rho_x\rho_y = 0, \qquad C\rho\rho_{yy} + D\rho_y^2 = 0, \\ A = 2(\omega - 1)(\alpha U + V), \qquad B = ((3 - 2\omega)\alpha - 1/\alpha)U + 2(2 - \omega)V, \\ C = 2(\omega - 1)(\alpha V - U), \qquad D = 2(\omega - 2)U + ((3 - 2\omega)\alpha - 1/\alpha)V. \end{gathered} \tag{3.9}$$

The compatibility conditions of system (3.9) are

$$(U^2 + V^2)\rho_x = 0, \qquad (U^2 + V^2)\rho_y = 0.$$

For $\rho_x = \rho_y = 0$, we have $u = u(t)$, $v = v(t)$, $\rho = \rho(t)$, and $T = T(t)$, i.e., the solution is invariant with respect to the subgroup $\{X_1, X_2\}$. For $U^2 + V^2 = 0$, we obtain a quiescent state: $u = v = 0$. If $\omega \neq 0$, system (1.1)–(1.4) yields $\rho \equiv \text{const}$ and $T \equiv \text{const}$, i.e., the solution is invariant with respect to the subgroup $\{X_1, X_2\}$. For $\omega = 0$, the solution of system (1.1)–(1.4) is recovered from the solution $\tau(x, y)$ of the equation

$$\varkappa(\tau_{xx} + \tau_{yy}) = c_V q$$

by the formulas

$$u = 0, \qquad v = 0, \qquad \rho = \tau^{-1}, \qquad T = T_0 \mathrm{e}^{qt}\tau, \qquad q \equiv \text{const.}$$

Reduction to a solution invariant with respect to some two-dimensional subgroup of the group $\{X_1, X_2, X_5 + \alpha X_8\}$ can be performed only in two cases. In the first, trivial, case ($\tau \equiv \text{const}$ and $q = 0$), the solution is invariant with respect to the subgroup $\{X_1, X_2\}$. In the second case with

$$\tau = \frac{c_V q}{4\varkappa}(x^2 + y^2) + a_{10}x + a_{01}y + \frac{\varkappa}{c_V q}(a_{10}^2 + a_{01}^2), \qquad q \neq 0, \quad a_{10}^2 + a_{01}^2 \neq 0,$$

the solution is reducible to a solution invariant with respect to the subgroup $\{aX_1 + X_5 + \alpha X_8, bX_1 + X_5 + \alpha X_8\}$, where $a = (2\varkappa/(c_V q))(a_{01} + \alpha a_{10})$ and $b = (2\varkappa/(c_V q))(\alpha a_{01} - a_{10})$. In all other cases, the solution is nonreducible. (It should be noted that, in the case $a_{10}^2 + a_{01}^2 \neq 0$, the solution is invariant with respect to the subgroup $\{X_5 + \alpha X_8\}$ but no reduction occurs, because the rank of the solution increases.)

*Subgroup* 5 ($\omega = 1$). The solution is presented as

$$\begin{gathered} u = T^{1/2}\Big(\varphi(t)\sin\frac{\ln T}{2\alpha} + \psi(t)\cos\frac{\ln T}{2\alpha}\Big) \equiv U(t, T), \\ v = T^{1/2}\Big(\varphi(t)\cos\frac{\ln T}{2\alpha} - \psi(t)\sin\frac{\ln T}{2\alpha}\Big) \equiv V(t, T), \\ \rho = \rho(t), \qquad T = T(t, x, y). \end{gathered}$$

We have

$$\frac{\partial U}{\partial T} = \frac{\alpha U + V}{2\alpha T}, \qquad \frac{\partial V}{\partial T} = \frac{\alpha V - U}{2\alpha T}.$$

The condition of velocity-field linearity takes the form

$$\begin{gathered} ATT_{xx} + BT_x^2 = 0, \qquad ATT_{xy} + BT_xT_y = 0, \qquad ATT_{yy} + BT_y^2 = 0, \\ CTT_{xx} + DT_x^2 = 0, \qquad CTT_{xy} + DT_xT_y = 0, \qquad CTT_{yy} + DT_y^2 = 0, \\ A = 2\alpha(\alpha U + V), \qquad B = -(\alpha^2 + 1)U, \\ C = 2\alpha(\alpha V - U), \qquad D = -(\alpha^2 + 1)V. \end{gathered} \tag{3.10}$$

The compatibility conditions of system (3.10) are

$$(U^2+V^2)T_x=0, \qquad (U^2+V^2)T_y=0.$$

In this case, all solutions of system (1.1)–(1.4) are invariant with respect to the subgroup $\{X_1, X_2\}$.

*Subgroup* 6 ($\omega \neq 1$). The solution is presented as

$$u=\frac{1}{t}\,\rho^{1/(2\omega-2)}\Big(\varphi(t)\sin\frac{\ln\rho}{2\alpha(\omega-1)}+\psi(t)\cos\frac{\ln\rho}{2\alpha(\omega-1)}\Big)+\frac{x}{t}\equiv U(t,x,\rho),$$

$$v=\frac{1}{t}\,\rho^{1/(2\omega-2)}\Big(\varphi(t)\cos\frac{\ln\rho}{2\alpha(\omega-1)}-\psi(t)\sin\frac{\ln\rho}{2\alpha(\omega-1)}\Big)+\frac{y}{t}\equiv V(t,y,\rho),$$

$$T=T_1(t)\rho^{1/(\omega-1)}, \qquad \rho=\rho(t,x,y).$$

We have

$$\frac{\partial U}{\partial\rho}=\frac{t(\alpha U+V)-(\alpha x+y)}{2\alpha(\omega-1)t\rho}, \qquad \frac{\partial V}{\partial\rho}=\frac{t(\alpha V-U)-(\alpha y-x)}{2\alpha(\omega-1)t\rho}.$$

The condition of velocity-field linearity reduces to the system

$$\begin{gathered} A\rho\rho_{xx}+B\rho_x^2=0, \qquad A\rho\rho_{xy}+B\rho_x\rho_y=0, \qquad A\rho\rho_{yy}+B\rho_y^2=0, \\ C\rho\rho_{xx}+D\rho_x^2=0, \qquad C\rho\rho_{xy}+D\rho_x\rho_y=0, \qquad C\rho\rho_{yy}+D\rho_y^2=0, \\ A=(\alpha U+V)t-(\alpha x+y), \qquad C=(\alpha V-U)t-(\alpha y-x), \\ B=\frac{(\alpha^2-2\alpha(\omega-1)-1)(Ut-x)+2(\alpha-\omega+1)(Vt-y)}{2(\omega-1)}, \\ D=\frac{(\alpha^2-2\alpha(\omega-1)-1)(Vt-y)-2(\alpha-\omega+1)(Ut-x)}{2(\omega-1)}. \end{gathered} \tag{3.11}$$

The compatibility conditions of system (3.11) are

$$((Ut-x)^2+(Vt-y)^2)\rho_x=0, \qquad ((Ut-x)^2+(Vt-y)^2)\rho_y=0.$$

For $\rho_x=\rho_y=0$, we have

$$u=u_1(t)+x/t, \qquad v=v_1(t)+y/t, \qquad \rho=\rho(t), \qquad T=T(t),$$

i.e., the solution is invariant with respect to the subgroup $\{X_3, X_4\}$.

We consider the case $u=x/t$, $v=y/t$, $\rho=\rho(t,x,y)$, and $\rho_x^2+\rho_y^2\neq 0$. Equations (1.1)–(1.3) acquire the form

$$\omega(2(l_0+m_0)T_1^{\omega-1}-Rt)=0, \qquad t\rho_t+x\rho_x+y\rho_y+2\rho=0.$$

For $\omega=0$, the solution of system (1.1)–(1.4) is recovered from the solution $\Psi(\xi,\eta)$, $T_1(t)$ of the system

$$\Psi_{\xi\xi}+\Psi_{\eta\eta}=A, \qquad \frac{c_V(t^2T_1)'-4(l_0+m_0)}{t^2T_1}=Ak_0-\frac{2R}{t}, \qquad A\equiv\text{const}$$

by the formulas

$$u=\xi, \qquad v=\eta, \qquad \rho=\frac{1}{t^2\Psi}, \qquad T=t^2T_1\Psi, \qquad \xi=\frac{x}{t}, \qquad \eta=\frac{y}{t}.$$

Reduction to a solution invariant with respect to some two-dimensional subgroup of the group $\{X_3, X_4, X_5+\alpha X_8\}$ can be performed in two cases only. In the first, trivial, case ($\Psi\equiv\text{const}$ and $A=0$), the solution is invariant with respect to the subgroup $\{X_3, X_4\}$. In the second case with

$$\Psi=A(\xi^2+\eta^2)/4+a_{10}\xi+a_{01}\eta+2(a_{10}^2+a_{01}^2)/A, \qquad A\neq 0, \quad a_{10}^2+a_{01}^2\neq 0,$$

the solution is reduced to a solution invariant with respect to the subgroup $\{aX_3+X_5+\alpha X_8, bX_4+X_5+\alpha X_8\}$, where $a=2(a_{01}+\alpha a_{10})/A$ and $b=2(\alpha a_{01}-a_{10})/A$. No reduction occurs in all other cases.

For $\omega \neq 0$, there exist no solutions that satisfy condition (1.5).

*Subgroup* 6 ($\omega = 1$). The solution is presented as

$$u = \frac{1}{t}T^{1/2}\Big(\varphi(t)\sin\frac{\ln T}{2\alpha} + \psi(t)\cos\frac{\ln T}{2\alpha}\Big) + \frac{x}{t} \equiv U(t,x,T),$$

$$v = \frac{1}{t}T^{1/2}\Big(\varphi(t)\cos\frac{\ln T}{2\alpha} - \psi(t)\sin\frac{\ln T}{2\alpha}\Big) + \frac{y}{t} \equiv V(t,y,T),$$

$$\rho = \rho(t), \qquad T = T(t,x,y).$$

We have

$$\frac{\partial U}{\partial T} = \frac{\alpha(Ut-x)+(Vt-y)}{2\alpha tT}, \qquad \frac{\partial V}{\partial T} = \frac{\alpha(Vt-y)-(Ut-x)}{2\alpha tT}.$$

The condition of velocity-field linearity reduces to the system

$$\begin{gathered}
ATT_{xx} + BT_x^2 = 0, \qquad ATT_{xy} + BT_xT_y = 0, \qquad ATT_{yy} + BT_y^2 = 0,\\
CTT_{xx} + DT_x^2 = 0, \qquad CTT_{xy} + DT_xT_y = 0, \qquad CTT_{yy} + DT_y^2 = 0,\\
A = 2(\alpha(Ut-x)+(Vt-y)), \qquad B = -(\alpha+1/\alpha)(Ut-x),\\
C = 2(\alpha(Vt-y)-(Ut-x)), \qquad D = -(\alpha+1/\alpha)(Vt-y).
\end{gathered} \tag{3.12}$$

The compatibility conditions of system (3.12) are

$$((Ut-x)^2+(Vt-y)^2)T_x = 0, \qquad ((Ut-x)^2+(Vt-y)^2)T_y = 0.$$

In this case, the only solution of system (1.1)–(1.4) that satisfies condition (1.5) has the form

$$u = u_1(t) + x/t, \qquad v = v_1(t) + y/t, \qquad \rho = \rho(t), \qquad T = T(t),$$

i.e., the solution is invariant with respect to the subgroup $\{X_3, X_4\}$.

*Subgroup* 7. The solution is presented as

$$u = u(t,x,y), \qquad v = uv_1(y), \qquad \rho = u^{2\omega-1}\rho_1(y), \qquad T = u^2T_1(y).$$

An analysis of velocity-field linearity yields one of the three presentations of the velocity-vector components:

1) $u = u_1(t)x + u_2(t)y + u_3(t)$, $\quad v = v_0(u_1(t)x + u_2(t)y + u_3(t))$;
2) $u = (y+u_0)u_1(t)$, $\quad v = (v_0 + v_{01}(y+u_0))u_1$;
3) $u = u(t)$, $\quad v = (v_0y + v_{01})u$.

In case 1, all solutions either are a subset of solutions in case 2 or do not satisfy condition (1.5). In case 2, all solutions are reducible to solutions invariant with respect to the subgroup $\{X_1, \alpha X_6 + \beta(X_7 - X_8)\}$. In case 3, the solution is reducible to a solution invariant with respect to the subgroup $\{X_1, X_6\}$.

*Subgroup* 8. The solution is presented as

$$\frac{ux+vy}{vx-uy} = Q(\xi), \qquad \rho = \frac{(u^2+v^2)^{\omega-1/2}}{\sqrt{x^2+y^2}}\rho_1(\xi), \qquad T = \frac{(ux+vy)^2}{x^2+y^2}T_1(\xi),$$

$$\xi = \sqrt{x^2+y^2}\exp(\alpha\arctan(y/x)).$$

As the function $Q(\xi)$ is independent of $t$, we can readily obtain the presentation for the velocity components in the case of a linear velocity field:

$$u = (u_{01}x + u_{02}y + u_{03})w(t), \qquad v = (v_{01}x + v_{02}y + v_{03})w(t).$$

A further analysis shows that Eq. (1.4) takes the form

$$\frac{dw}{dt} + F(x, y, \xi)w^2 = 0,$$

where $F$ is a known function. Integration of the last equation yields the presentation for the function $w$

$$w(t) = 1/(w_0 t + w_{01}).$$

The solution obtained is reducible to a solution invariant with respect to the subgroup $\{X_5 + \alpha X_8, w_{01} X_6 + w_0(X_7 - X_8)\}$.

**Conclusions.** Thus, we proved a sufficient condition of reducibility of partially invariant solutions of rank 1 and defect 1 of equations of plane motion of a viscous heat-conducting perfect gas. It is also shown that the solutions of system (1.1)–(1.4) with uniform deformation contain only two partially invariant solutions of rank 1 and defect 1 that are not reducible to invariant solutions. The first solution is of no special interest because it describes the distribution of thermodynamic parameters in a quiescent gas. The second solution is recovered from the solution of the Poisson equation. The initial-boundary problems for system (1.1)–(1.4) are easily reduced to a boundary-value problem for the Poisson equation. Setting the velocity at the boundary has to be correlated with the solution. Setting the temperature at the domain boundary corresponds to the Dirichlet problem, and setting the flux of temperature corresponds to the Neumann problem; mixed problems are also possible. Elliptic equations in the domain with a curvilinear boundary can be solved numerically with extremely high accuracy [12]. Therefore, the new solution of system (1.1)–(1.4) is obtained numerically with arbitrary required accuracy and can be used as a test for formulas, algorithms, and their program implementation in development of numerical methods and computational codes.

It should be noted that the new solutions are principally different from the known solutions with a linear dependence of the velocity vector on some of the spatial coordinates, which were obtained in [11] for equations of dynamics of a viscous incompressible heat-conducting fluid and isentropic flows of a compressible gas with a polytropic equation of state. Previously, all thermodynamic parameters (temperature, pressure, entropy, etc.) in all solutions depended linearly or quadratically on some of the spatial coordinates. These restrictions were not imposed in the present work. As a result, we managed to obtain solutions where the dependence of temperature on the spatial coordinates is more complicated than a polynomial curve (described by solving the Poisson equations). Solutions with a linear dependence of temperature on the spatial coordinates are obtained as a particular case of new solutions being constructed; it is also proved that all of them (as well as solutions with a quadratic dependence of temperature on the spatial coordinates) are reducible to invariant solutions.

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